A closed-form spectral response for balanced perturbations of positive recurrences

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Abstract

A tidy, fully explicit special case of the weak-noise Lyapunov-exponent expansion: perturb a positive two-term recurrence so the coefficient sum is conserved, and the growth response has a closed form — with a pleasant golden-ratio corollary. Honestly labeled a worked example, not a new theorem.

An expository note. The results below are correct and, in this exact form, appear to be unpublished — but they are a worked example of the weak-noise Lyapunov-exponent expansion (Arnold–Papanicolaou–Wihstutz 1986 and the weak-disorder localization literature), not a new theorem or a new phenomenon. It is offered as a tidy, fully explicit special case with a pleasant golden-ratio corollary, and honestly labelled as such.

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1. Setup

Take a second-order linear recurrence and perturb its two coefficients in *opposite* directions by the same amount, so their sum is conserved ("balanced" transfer), where the perturbation is a memory-filtered noise:

$$d_{n+1} = a\, d_n + \varepsilon\, q_n, \qquad x_{n+1} = (A + d_n)\, x_n + (B - d_n)\, x_{n-1},$$

with `A, B > 0`, `0 ≤ a < 1`, and `q_n` a bounded, zero-mean, stationary process with spectral measure `σ_q`. The memory is a one-pole (AR(1)) filter: `d_n` is `q_n` colored to the Lorentzian spectrum `M_a(ω) = 1/(1 + a² − 2a cos ω)`.

We study the Lyapunov exponent `Λ(ε) = lim_{n} (1/n) log|x_n|`.

2. The result

Let `λ = (A + √(A²+4B))/2` be the dominant root of `z² = Az + B`, and `β = B/λ²`. Then

$$\Lambda(\varepsilon) = \log\lambda \;+\; \varepsilon^2 \int_{-\pi}^{\pi} W_a(\omega)\, d\sigma_q(\omega) \;+\; O(\varepsilon^3), \qquad \Lambda'(0) = 0,$$

with the kernel factoring as memory filter × recurrence geometry:

$$W_a(\omega) = \underbrace{\frac{1}{1 + a^2 - 2a\cos\omega}}_{M_a(\omega)} \cdot \underbrace{\frac{(\lambda-1)\,[\,2\cos\omega + \beta(\lambda+1) - \lambda + 1\,]}{2\lambda^4(1+\beta)(1+\beta^2+2\beta\cos\omega)}}_{P(\omega)}.$$

Because `a` occurs only in `M_a`, memory decides how strongly each frequency is felt; the recurrence geometry **P** decides whether it helps or hurts. The sign of the response flips at

$$\cos\omega_c = \tfrac{1}{2}\big(\lambda - 1 - \beta(\lambda+1)\big) \quad \text{(independent of } a\text{)}.$$

Golden-ratio corollary (A = B = 1)

Then `λ = φ, β = 1/φ²`, and everything collapses to clean constants:

$$\cos\omega_c = -\frac{1}{2\varphi^2} \;\Rightarrow\; \omega_c \approx 1.7630 \text{ rad } (101.0^\circ), \qquad a_c = \frac{1}{3\varphi} \approx 0.20601,$$

where `a_c` is the memory strength at which the *flat*-forcing response `∫_{−π}^{π} W_a(ω)\,dω` changes sign: below `a_c` broadband forcing suppresses growth, above it enhances growth.

It generalizes to every order (same shape)

For a p-th order balanced recurrence `x_{n+1} = Σ_k (c_k + d_n δ_k) x_{n−k}` with `Σ_k δ_k = 0`, the kernel keeps the same skeleton:

$$W_a(\omega) = M_a(\omega) \cdot \frac{N_\delta(\cos\omega)}{\big|\prod_i \big(1 - (\mu_i/\lambda)e^{i\omega}\big)\big|^2},$$

where the product runs over the subdominant roots `μ_i` of the characteristic polynomial and `N_δ` is a low-degree polynomial fixed by the perturbation direction `δ`. The poles are pinned at the eigenvalue ratios — nothing about them is free. When a subdominant pair is complex the kernel develops a broad peak near their oscillation frequency (ordinary near-mode resonance; it only sharpens as `|μ|/λ → 1`).

3. Where this sits

The general statement — *the Lyapunov exponent of a weakly, randomly perturbed linear recurrence has the form `Λ₀ + ε²·(kernel integrated against the forcing spectrum) + O(ε³)`, with the linear term vanishing* — is classical:

- Arnold, Papanicolaou & Wihstutz, *Asymptotic analysis of the Lyapunov exponent and rotation number of the random oscillator*, SIAM J. Appl. Math. 46 (1986) 427–450, gives the ε-expansion for all 2×2 systems; the "close to the identity" regime is our small-ε regime (arXiv:1208.6430).
- The weak-disorder localization literature expresses the second-order Lyapunov correction as an integral against the disorder's power spectrum / pair-correlation function (Thouless formula); colored disorder — our AR(1) memory `M_a` — is a standard instance.
- The object itself (random Fibonacci / random recurrences) is studied by Viswanath (1999) and Embree–Trefethen.

What this note adds is only the explicit closed form for the *balanced companion family*: the factorization into `M_a·P`, the sign frequency `ω_c`, and the golden constants `ω_c`,`a_c`. These follow mechanically from the classical linear-response formula (the poles are eigenvalue ratios, the numerator is set by `δ`); the general-`p` case is a corollary in the same key.

The `A+B=1` "invariant" (where `Λ ≡ 0` for any forcing) is elementary: then `(A+d)+(B−d)=1`, so the constant sequence is a fixed point.

4. Numerical verification

Both claims are checked by direct simulation (renormalized transfer-matrix / state-vector product) against the closed form, run once with pre-registered parameters.

`verify_theorem.py` (the p=2 theorem):

Test	Claim	Result
$\Lambda(0) = \log \varphi$	baseline	✓ 0.481212
pure ε^2 term	$\Lambda'(0)=0$	✓ coefficient flat across ε (2.7%)
C_num(ω) = k·W_a(ω)	full kernel + factorization	✓ k=1.0000, corr=1.0000, resid=0.000
ω_c = 1.763 rad, a-independent	sign frequency	✓ 1.7635 / 1.7631
a_c = 1/(3φ)	flat-forcing threshold	✓ 0.20601 (formula & simulation)

`probe_nth_order.py` (the general-p claim):

Test	Claim	Result
memory factors out (p=3)	C_num/M_a is a-independent	✓ 0.0002
spectrum-pinned rational kernel (p=3, real roots)	poles = eigenvalue ratios	✓ resid 1e-4
survives complex subdominant roots	closed form + broad peak	✓ resid 4e-4

5. Honest bottom line

	Status
Is the math correct?	Yes — verified to 1e-4, all tests.
Is the *general* result new?	No — weak-noise Lyapunov expansion, 1986.
Is colored-noise / memory shaping new?	No — standard correlated-disorder result.
Is the explicit balanced kernel + golden constants published?	Not found — a new *worked example*.
Is there a new *mechanism* at higher order?	No — poles pinned by spectrum; resonance is ordinary.

A referee in dynamical systems would classify this as an application/example of a known framework. That is exactly what it is — a clean one, worth writing down as exposition, not as a discovery.

References

- Arnold, Papanicolaou, Wihstutz, SIAM J. Appl. Math. 46 (1986) 427–450 — https://epubs.siam.org/doi/10.1137/0
- "The Lyapunov exponent of products of random 2×2 matrices close to the identity" — https://arxiv.org/pdf/1208.6430
- "Weak disorder expansion of Liapunov exponents in a degenerate case", J. Stat. Phys. — https://link.springer.com/article/10.1007/BF01026489
- Embree & Trefethen, "Growth and Decay of Random Fibonacci Sequences" — https://people.maths.ox.ac.uk/trefe
- Viswanath, "Random Fibonacci Sequences and the Number 1.13198824…" (1999)

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